

$\mathbb{R}^n \rtimes G(n)$ is Algebraically Determined

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Abstract

Let G be a Polish (i.e., complete separable metric topological) group. Define G to be an algebraically determined Polish group if for any Polish group L and algebraic isomorphism $\varphi : L \rightarrow G$, we have that φ is a topological isomorphism. Let $M(n, \mathbb{R})$ be the set of $n \times n$ matrices with real coefficients and let the group G in the above definition be the natural semidirect product $\mathbb{R}^n \rtimes G(n)$, where $n \geq 2$ and $G(n)$ is one of the following groups: either the general linear group $GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) \mid \det(A) \neq 0\}$, or the special linear group $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det(A) = 1\}$, or $|SL(n, \mathbb{R})| = \{A \in GL(n, \mathbb{R}) \mid |\det(A)| = 1\}$ or $GL^+(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det(A) > 0\}$. These groups are of fundamental importance for linear algebra and geometry. The purpose of this paper is to prove that the natural semidirect product $\mathbb{R}^n \rtimes G(n)$ is an algebraically determined Polish group. Such a result is not true for $\mathbb{C}^n \rtimes GL(n, \mathbb{C})$ nor even for $\mathbb{R}^3 \rtimes SO(3, \mathbb{R})$. The proof of this result is done in a sequence of steps designed to verify the hypotheses of the road map Theorem 2. A key intermediate result is that $\varphi^{-1}(SO(n, \mathbb{R}))$ is an analytic subgroup of L for every $n \geq 2$.

1 Introduction

Let G be a Polish (i.e., complete separable metric topological) group. Define G to be an algebraically determined Polish group if for any Polish group L and algebraic isomorphism $\varphi : L \mapsto G$, we have that φ is a topological isomorphism. Note that if G is algebraically determined then every algebraic automorphism of G is continuous.

Let $M(n, \mathbb{R})$ be the set of $n \times n$ matrices with real coefficients and let the group G in the above definition be the natural semidirect product $\mathbb{R}^n \rtimes G(n)$, where $n \geq 2$ and $G(n)$ is one of the following groups: either the general linear group $GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) \mid \det(A) \neq 0\}$, or the special linear group $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det(A) = 1\}$, or $|SL(n, \mathbb{R})| = \{A \in GL(n, \mathbb{R}) \mid |\det(A)| = 1\}$ or $GL^+(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det(A) > 0\}$. These groups are of fundamental importance for linear algebra and geometry. The purpose of this paper is to prove the following theorem.

Theorem 1 .

Let L be a Polish group and let $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ be an algebraic isomorphism. Then φ is a topological isomorphism and therefore $\mathbb{R}^n \rtimes G(n)$ is an algebraically determined Polish group.

The only precedent for this theorem seems to be a result for the $ax + b$ group ([5]). The proof of Theorem 1 of necessity must be rather delicate, for the analogous result is not true for $(\mathbb{R}, +)$ ($(\mathbb{R}, +) \simeq (\mathbb{R}^2, +)$) nor for $\mathbb{C}^n \rtimes GL(n, \mathbb{C})$ (the field \mathbb{C} has $2^{2^{\aleph_0}}$ discontinuous automorphisms) nor even for $\mathbb{R}^3 \rtimes SO(3, \mathbb{R})$ ([11]).

Throughout this paper we will make free use of basic facts, terminology and notation in descriptive set theory ([3], [6], [8], [9], [10]). In particular, if X is a Polish space then $\mathcal{BP}(X)$ denotes the σ -algebra of subsets of X with the Baire property, i.e., the σ -algebra generated by the Borel subsets and first category subsets of X . Analytic sets are sets with the Baire property.

Kechris and Rosendal [7] introduced the notion of Polish groups with ample generics. This is an important but very special class of Polish groups since any homomorphism of such a Polish group into a separable topological group is automatically continuous. Obviously any Polish group with ample generics is algebraically determined. The converse is false since it is easy to see that any nontrivial connected Lie group cannot have ample generics even though there are many examples of Lie groups (e.g., the real $ax + b$ group) that are algebraically determined.

The proof of Theorem 1 is done in a sequence of steps following the road map given in the following theorem.

Theorem 2 ([1], Theorem 4).

Let K and Q be two Polish groups and let $\theta : Q \mapsto \text{Aut}(K)$ be a group homomorphism that satisfies $K \times Q \mapsto K$, $(k, q) \mapsto \theta_q(k)$ is continuous. Then $K \rtimes_\theta Q$ is a Polish group in the product topology. Let L be a Polish group and let $\varphi : L \mapsto K \rtimes_\theta Q$ be a group isomorphism. If $\varphi^{-1}(K)$ and $\varphi^{-1}(Q)$ are both analytic subgroups of L , then both $\varphi^{-1}(K)$ and $\varphi^{-1}(Q)$ are closed subgroups of L . Next, if, in addition, $\varphi|_{\varphi^{-1}(K)} : \varphi^{-1}(K) \mapsto K$ is measurable with respect to $\mathcal{BP}(\varphi^{-1}(K))$, then $\varphi|_{\varphi^{-1}(K)}$ is a topological isomorphism. Furthermore, if, in addition, θ is injective, then $\varphi|_{\varphi^{-1}(Q)} : \varphi^{-1}(Q) \mapsto Q$ is a topological isomorphism. Finally, under all of these conditions, $\varphi : L \mapsto K \rtimes_\theta Q$ is a topological isomorphism and thus $K \rtimes_\theta Q$ is an algebraically determined Polish group.

The two steps that are usually the most difficult to verify is that $\varphi^{-1}(Q)$ is an analytic subgroup of L and that $\varphi|_{\varphi^{-1}(K)} : \varphi^{-1}(K) \mapsto K$ is measurable with respect to $\mathcal{BP}(\varphi^{-1}(K))$. We attack these problems in the following sections using the notation of Theorem 1. Section 2 is devoted to proving that $\varphi^{-1}(SL(n, \mathbb{R}))$ is an analytic subgroup of L as well noting a few other simple but important observations.

2 $\varphi^{-1}(SL(n, \mathbb{R}))$ is an Analytic Subgroup of L

First note that each natural semidirect product $\mathbb{R}^n \rtimes G(n)$ is a Polish topological group.

Lemma 3 .

$\mathbb{R}^n \simeq \{(x, I_n) \mid x \in \mathbb{R}^n\}$ is a maximal abelian subgroup of $\mathbb{R}^n \rtimes G(n)$. If L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(\mathbb{R}^n)$ is a closed normal maximal abelian subgroup of L .

Proof:

A simple computation shows that \mathbb{R}^n is maximal abelian subgroup of $\mathbb{R}^n \rtimes G(n)$. Hence, $\varphi^{-1}(\mathbb{R}^n)$ is closed subgroup of L since any maximal abelian subgroup of a Hausdorff topological group is closed. Therefore $\varphi^{-1}(\mathbb{R}^n)$ is a closed normal abelian subgroup of L . \square

Next notice that $SL(n, \mathbb{R}) \subseteq G(n) \subseteq \mathbb{R}^n \rtimes G(n)$ for each $n \geq 2$. The rest of this section is devoted to the proof that $\varphi^{-1}(SL(n, \mathbb{R}))$ is an analytic subgroup of L for each $n \geq 2$. The two cases n is even or n is odd will be discussed separately.

Lemma 4 .

If G is a Polish group and $\emptyset \neq A \subseteq G$ is an analytic set, then $[A, A]$, the subgroup of G generated by commutators of pairs of elements of A , is an analytic subgroup of G . If $A, B \subseteq G$ are nonempty analytic subsets of G , then the subgroup of G algebraically generated by A and B is an analytic subgroup of G .

Proof:

This is elementary using the basic facts that a product of two analytic sets is analytic, that the continuous image of an analytic set is an analytic set and that the union of a sequence of analytic sets is analytic. \square

Proposition 5 .

If $-I_n \in G(n)$, then the centralizer of $(0, -I_n)$ in $\mathbb{R}^n \rtimes G(n)$ is $\{(0, A) \mid A \in G(n)\} \simeq G(n)$. This is true if $n \geq 2$ is even, if $G(n) = GL(n, \mathbb{R})$ or if $G(n) = |SL(n, \mathbb{R})|$. For such $G(n)$, if L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(G(n))$ is a closed subgroup of L . In addition under these circumstances $\varphi^{-1}(SL(n, \mathbb{R}))$ is an analytic subgroup of L .

Proof :

It is an elementary calculation that the centralizer of $(0, -I_n)$ in $\mathbb{R}^n \rtimes G(n)$ is $\{(0, A) \mid A \in G(n)\} \simeq G(n)$. Also,

$$\begin{aligned} \varphi^{-1}(G(n)) &= \varphi^{-1}(\{(0, A) \mid A \in G(n)\}) \\ &= \varphi^{-1}(\text{the centralizer of } (0, -I_n) \text{ in } \mathbb{R}^n \rtimes G(n)) \\ &= \text{the centralizer of } \varphi^{-1}((0, -I_n)) \text{ in } L \end{aligned}$$

Thus $\varphi^{-1}(G(n))$ is closed in L .

Now using the fact that

$$\begin{aligned} SL(n, \mathbb{R}) &= [SL(n, \mathbb{R}), SL(n, \mathbb{R})] \\ &= [GL(n, \mathbb{R}), GL(n, \mathbb{R})] \end{aligned}$$

([4], section 6.7, Lemma 2, page 377) we obtain that

$$\begin{aligned} SL(n, \mathbb{R}) &= [SL(n, \mathbb{R}), SL(n, \mathbb{R})] \\ &\subseteq [G(n), G(n)] \\ &\subseteq [GL(n, \mathbb{R}), GL(n, \mathbb{R})] \\ &= SL(n, \mathbb{R}) \end{aligned}$$

Therefore $\varphi^{-1}(SL(n, \mathbb{R})) = \varphi^{-1}([G(n), G(n)]) = [\varphi^{-1}(G(n)), \varphi^{-1}(G(n))]$ is an analytic subgroup of L by Lemma 4 since $\varphi^{-1}(G(n))$ is closed in L . \square

The analogous result in case n is odd is much more delicate.

Lemma 6 .

If $n \geq 3$ is odd, then the centralizer of $\left(0, \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}\right)$ in $\mathbb{R}^n \rtimes G(n)$ is

$$\mathcal{A} = \left\{ \left(\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & A \end{pmatrix} \right) \in \mathbb{R}^n \rtimes G(n) \right\}, \text{ where } \lambda \neq 0, \lambda \text{ and } x_1 \in \mathbb{R}, A \in GL(n-1) \text{ with either } \lambda \cdot \det(A) \neq$$

0 if $\mathbb{R}^n \rtimes G(n) = \mathbb{R}^n \rtimes GL(n, \mathbb{R})$, or $\lambda \cdot \det(A) > 0$ if $\mathbb{R}^n \rtimes G(n) = \mathbb{R}^n \rtimes GL^+(n, \mathbb{R})$, or $\lambda \cdot \det(A) = 1$ if $\mathbb{R}^n \rtimes G(n) = \mathbb{R}^n \rtimes SL(n, \mathbb{R})$ or $|\lambda \cdot \det(A)| = 1$ if $\mathbb{R}^n \rtimes G(n) = \mathbb{R}^n \rtimes |SL(n, \mathbb{R})|$. If L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(\mathcal{A})$ is a closed subgroup of L .

Proof:

If $\left(x, \begin{pmatrix} \lambda & \gamma \\ \beta & A \end{pmatrix}\right) \in \mathbb{R}^n \rtimes G(n)$, then

$$\begin{aligned} \left(x, \begin{pmatrix} \lambda & \gamma \\ \beta & A \end{pmatrix}\right) \cdot \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}\right) &= \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}\right) \cdot \left(x, \begin{pmatrix} \lambda & \gamma \\ \beta & A \end{pmatrix}\right) \\ \text{if and only if } \left(x, \begin{pmatrix} \lambda & \gamma \\ \beta & A \end{pmatrix}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix} \cdot \left(x, \begin{pmatrix} \lambda & \gamma \\ \beta & A \end{pmatrix}\right) \\ \text{if and only if } \left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} \lambda & -\gamma \\ \beta & -A \end{pmatrix}\right) &= \left(\begin{pmatrix} x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix}, \begin{pmatrix} \lambda & \gamma \\ -\beta & -A \end{pmatrix}\right). \end{aligned}$$

Therefore $\beta = 0$, $\gamma = 0$ and $x_i = 0$ for $2 \leq i \leq n$ and $\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} \lambda & \gamma \\ \beta & A \end{pmatrix}\right) = \left(\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & A \end{pmatrix}\right)$

with $\lambda, x_1 \in \mathbb{R}$, $A \in GL(n-1)$ and $\lambda \cdot \det(A)$ is any one of the four mentioned cases.

Now, $\varphi^{-1}(\mathcal{A}) = \varphi^{-1}(\text{the centralizer of } \begin{pmatrix} 0, \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix} \end{pmatrix} \text{ in } \mathbb{R}^n \rtimes G(n)) = \text{the centralizer of } \varphi^{-1}((0, \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix})) \text{ in } L$, which is a closed subgroup of L . \square

Lemma 7 .

If $n \geq 3$ is odd and \mathcal{A} is as in Lemma 6, then in all four cases the commutator subgroup of \mathcal{A} is

$$\mathcal{D} = \left\{ \left(\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \right) \mid x_1 \in \mathbb{R}, D \in SL(n-1, \mathbb{R}) \right\}. \text{ If } L \text{ is a Polish group and } \varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$$

is an algebraic isomorphism, then $\varphi^{-1}(\mathcal{D})$ is an analytic subgroup of L .

Proof:

$$\text{Compute that } \left[\left(\begin{pmatrix} x' \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda' & 0 \\ 0 & A' \end{pmatrix} \right), \left(\begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & A \end{pmatrix} \right) \right] = \left(\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \right),$$

where $x_1 = (-1 + \lambda')x + (1 - \lambda)x'$ and $A'AA'^{-1}A^{-1} = D \in SL(n-1, \mathbb{R})$.

On the other hand elements of the form $\left(\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \right)$ are in the commutator subgroup of \mathcal{A} since

if we take $\lambda' = 1/2$, $\lambda = 1/2$, $x = -x_1$, $x' = x_1$ and A and A' suitable simple diagonal matrices (remember $n \geq 3$), then $\left(\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix} \right)$ is in the commutator subgroup of \mathcal{A} . Also, $\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \right)$, where

$D \in SL(n-1, \mathbb{R})$, is in the commutator subgroup of \mathcal{A} ([4]). Therefore, the product

$$\left(\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \right) \text{ is in the commutator subgroup of } \mathcal{A}.$$

Finally, $\varphi^{-1}(\mathcal{D}) = \varphi^{-1}(\text{the commutator subgroup of } \mathcal{A}) = \text{the commutator subgroup of } \varphi^{-1}(\mathcal{A})$, which is an analytic subgroup of L by Lemma 4. \square

Lemma 8 .

$$\text{If } n \geq 3 \text{ is odd, then the commutator subgroup of } \mathcal{D} = \left\{ \left(\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \right) \mid x_1 \in \mathbb{R}, D \in SL(n-1, \mathbb{R}) \right\}$$

is $\mathcal{B} = \left\{ \left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \right) \mid B \in SL(n-1, \mathbb{R}) \right\}$ and $\varphi^{-1}(\mathcal{B})$ is an analytic subgroup of L .

Also the commutator subgroup of $\mathcal{D}' = \left\{ \left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \right) \mid x_n \in \mathbb{R}, D \in SL(n-1, \mathbb{R}) \right\}$ is

$\mathcal{B}' = \left\{ \left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \right) \mid B \in SL(n-1, \mathbb{R}) \right\}$ and $\varphi^{-1}(\mathcal{B}')$ is an analytic subgroup of L .

Proof:

Compute that $\left[\left(\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \right), \left(\begin{pmatrix} x'_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & D' \end{pmatrix} \right) \right] = \left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \right)$, where

$B = DD'D^{-1}D'^{-1} \in SL(n-1, \mathbb{R})$. On the other hand elements of the form $\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \right)$ are in the

commutator subgroup of \mathcal{D} by [4]. Now $\varphi^{-1}(\mathcal{B}) = \varphi^{-1}$ (the commutator subgroup of \mathcal{D}) = the commutator subgroup of $\varphi^{-1}(\mathcal{D})$, which is an analytic subgroup of L by Lemma 4.

Similarly one can show that the commutator subgroup of \mathcal{D}' is \mathcal{B}' and therefore $\varphi^{-1}(\mathcal{B}')$ is an analytic subgroup of L . \square

Lemma 9 .

If $n \geq 3$ is odd, let \mathcal{B} and \mathcal{B}' be as in Lemma 8 and let \mathcal{K} be the group generated by \mathcal{B} and \mathcal{B}' . Then $\varphi^{-1}(\mathcal{K})$ is an analytic subgroup of L .

Proof:

Both $\varphi^{-1}(\mathcal{B})$ and $\varphi^{-1}(\mathcal{B}')$ are analytic subgroups of L by Lemma 8. Therefore, the group generated by them, namely $\varphi^{-1}(\mathcal{K})$, is also an analytic subgroup of L by Lemma 4. \square

Lemma 10 .

If $n \geq 2$ view $\mathbb{R}^n - \{0\}$ as an $SL(n, \mathbb{R})$ -space and let $z = (1, 0, \dots, 0) \in \mathbb{R}^n - \{0\}$. Then the stability group $SL(n, \mathbb{R})_z = \{a \in SL(n, \mathbb{R}) \mid a \cdot z = z\}$ is contained in the group generated by elements of the form

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \mid B \in SL(n-1, \mathbb{R}) \right\}$ and $\left\{ \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \mid B \in SL(n-1, \mathbb{R}) \right\}$.

Proof:

If $\lambda_1, \lambda_2 \in \mathbb{R}$, compute that

$$\begin{pmatrix} 1 & 0 & \cdots & \lambda_1 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda_2 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \lambda_1 & \lambda_1 \lambda_2 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda_2 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 1 & 0 & \cdots & \lambda_1 & \lambda_1 \lambda_2 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda_2 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \cdots & -\lambda_1 & \lambda_1 \lambda_2 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\lambda_2 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \lambda_1 \lambda_2 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

An element $a \in SL(n, \mathbb{R})_z$ is of the form $\begin{pmatrix} 1 & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ 0 & & & & \\ \vdots & & A & & \\ 0 & & & & \end{pmatrix}$, where $A \in SL(n-1, \mathbb{R})$.

Observe that $\begin{pmatrix} 1 & a_{12} & \cdots & a_{1(n-1)} & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 & \lambda_1 \lambda_2 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1(n-1)} & \lambda_1 \lambda_2 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$

Hence $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} 1 & a_{12} & \cdots & a_{1(n-1)} & \lambda_1 \lambda_2 \\ 0 & & & & \\ \vdots & & I_{n-1} & & \\ 0 & & & & \end{pmatrix} = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1(n-1)} & \lambda_1 \lambda_2 \\ 0 & & & & \\ \vdots & & A & & \\ 0 & & & & \end{pmatrix}$, where

$A \in SL(n-1, \mathbb{R})$. So every element of the stability subgroup is in the subgroup generated by elements of the form $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \mid B \in SL(n-1, \mathbb{R}) \right\}$ and $\left\{ \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \mid B \in SL(n-1, \mathbb{R}) \right\}$. \square

Lemma 11 .

If $n \geq 2$, then $SL(n, \mathbb{R})$ acts transitively on $\mathbb{R}^n - \{0\}$ and $SL(n, \mathbb{R})/SL(n, \mathbb{R})_z \simeq \mathbb{R}^n - \{0\}$.

Proof:

This is well known and is an easy exercise. \square

Lemma 12 .

If $n \geq 3$ is odd, let \mathcal{K} be the subgroup of $SL(n, \mathbb{R})$ generated by $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \mid B \in SL(n-1, \mathbb{R}) \right\}$ and $\left\{ \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \mid B \in SL(n-1, \mathbb{R}) \right\}$. Then \mathcal{K} acts transitively on $\mathbb{R}^n - \{0\}$.

Proof:

Let $x = (x_1, x_2, \dots, x_n) \neq 0$ and $z = (1, 0, \dots, 0)$. It suffices to show there exists $g \in \mathcal{K}$ such that $g \cdot x = z$. Use Lemma 11. There exists $\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \in SL(n, \mathbb{R})$, where $B \in SL(n-1, \mathbb{R})$, such that

$$\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x'_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$
 Also $\begin{pmatrix} B' & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x'_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for some $B' \in SL(n-1, \mathbb{R})$. Therefore \mathcal{K} acts transitively on $\mathbb{R}^n - \{0\}$. \square

Lemma 13 .

If G is a group and $H \subset K \subset G$ are subgroups such that K acts transitively on G/H , then $K = G$.

Proof:

If $a \in G$, then $a \in aH$. Since K acts transitively on G/H , there exists $k \in K$ such that $a \in aH = k \cdot H \subset KH \subset K \cdot K = K$. Therefore $K = G$. \square

Proposition 14 .

If $n \geq 3$ is odd, then $SL(n, \mathbb{R})$ is the group generated by elements of the form $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \mid B \in SL(n-1, \mathbb{R}) \right\}$ and $\left\{ \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \mid B \in SL(n-1, \mathbb{R}) \right\}$. If L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(SL(n, \mathbb{R}))$ is an analytic subgroup of L .

Proof:

Let \mathcal{K} be the subgroup of $SL(n, \mathbb{R})$ generated by elements of the form $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \mid B \in SL(n-1, \mathbb{R}) \right\}$ and $\left\{ \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \mid B \in SL(n-1, \mathbb{R}) \right\}$. Then $SL(n, \mathbb{R})_z \subset \mathcal{K}$ by Lemma 10 and \mathcal{K} acts transitively on $SL(n, \mathbb{R})/SL(n, \mathbb{R})_z$ by Lemma 12. $SL(n, \mathbb{R}) = \mathcal{K}$ by Lemma 13. Therefore $\varphi^{-1}(SL(n, \mathbb{R})) = \varphi^{-1}(\mathcal{K})$ is an analytic subgroup of L by Lemma 9. \square

3 $\varphi^{-1}(G(n))$ is an Analytic Subgroup of L

The goal of this section is to prove that if L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(G(n))$ is an analytic subgroup of L . This has already been done in the cases $G(n) = GL(n, \mathbb{R})$ ($n \geq 2$ even), $G(n) = |SL(n, \mathbb{R})|$ ($n \geq 2$), $G(n) = SL(n, \mathbb{R})$ ($n \geq 2$) and $G(n) = GL^+(n, \mathbb{R})$ ($n \geq 2$ even). The only case remaining is $G(n) = GL^+(n, \mathbb{R})$ and $n \geq 3$ is odd. The results of the previous section on $SL(n, \mathbb{R})$ will lead to a quick proof. The following lemma certainly is well known.

Lemma 15 .

The centralizer of $SL(n, \mathbb{R})$ in $GL(n, \mathbb{R})$ is the group of nonzero scalar matrices.

Proof:

A scalar matrix will commute with any matrix in $GL(n, \mathbb{R})$. Conversely, let $A = (a_{ij})$ belong to the centralizer of $SL(n, \mathbb{R})$ in $GL(n, \mathbb{R})$. Write E_{ij} for the elementary $n \times n$ matrix with 1 in the (i, j) th position and 0 elsewhere. Now $I + E_{ij}$ is in $SL(n, \mathbb{R})$ if $i \neq j$ and so A and E_{ij} commute. The (k, j) th entry of AE_{ij} is a_{ki} while that of $E_{ij}A$ is 0 if $k \neq i$ and is a_{jj} if $k = i$. Hence $a_{ki} = 0$ if $k \neq i$ and $a_{ii} = a_{jj}$, which shows that A is a scalar. \square

Lemma 16 .

For $n \geq 2$ the centralizer of $SL(n, \mathbb{R})$ in $G(n)$, say C , is the set of diagonal elements of $G(n)$. If $G(n) = GL^+(n, \mathbb{R})$, then $GL^+(n, \mathbb{R}) = C \cdot SL(n, \mathbb{R})$.

Proof:

The first assertion follows from Lemma 15. For the second, suppose $T \in GL^+(n, \mathbb{R})$ and $\det(T) = \lambda^n > 0$. Let $D(\lambda)$ be the diagonal matrix with λ as the diagonal entries. $D(\lambda) \in C$ and $T = D(\lambda) \cdot D(\lambda)^{-1}T \in C \cdot SL(n, \mathbb{R})$. \square

Proposition 17 .

Let L be a Polish group and $\varphi : L \rightarrow \mathbb{R}^n \rtimes GL^+(n, \mathbb{R})$ be an algebraic isomorphism. Then $\varphi^{-1}(GL^+(n, \mathbb{R}))$ is an analytic subgroup of L .

Proof:

Let C be the centralizer of $SL(n, \mathbb{R})$ in $\mathbb{R}^n \rtimes GL^+(n, \mathbb{R})$. Then $C \subset GL^+(n, \mathbb{R})$, $GL^+(n, \mathbb{R}) = C \cdot SL(n, \mathbb{R})$ and $\varphi^{-1}(C)$ is closed in L since it is the centralizer of $\varphi^{-1}(SL(n, \mathbb{R}))$ in L . Hence, $\varphi^{-1}(GL^+(n, \mathbb{R})) = \varphi^{-1}(C \cdot SL(n, \mathbb{R})) = \varphi^{-1}(C) \cdot \varphi^{-1}(SL(n, \mathbb{R}))$ is a product of two analytic subsets of L and therefore itself is an analytic subset of L . \square

Corollary 18 .

If L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(G(n))$ is an analytic subgroup of L .

Proof:

This follows from the comments and results of this section. \square

4 $\varphi^{-1}(SO(n, \mathbb{R}))$ is an Analytic Subgroup of L

The goal of this section is to prove that if L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(SO(n, \mathbb{R}))$ is an analytic subgroup of L . The proof is factored into a sequence of steps.

Lemma 19 .

Let $n \geq 4$. The centralizer of $\mathcal{M} = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & M \end{pmatrix} \mid I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M \in SL(n-2, \mathbb{R}) \right\}$ in $SL(n, \mathbb{R})$ is

$$\mathcal{A} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid A \in GL(2, \mathbb{R}), D = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} \in GL(n-2, \mathbb{R}), \lambda \neq 0, \det(A) \cdot \det(D) = 1 \right\}.$$

If L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(\mathcal{A})$ is an analytic subgroup of L .

Proof:

That \mathcal{A} is as is stated follows from Lemma 15 and an elementary computation. Furthermore $\varphi^{-1}(\mathcal{A}) = \varphi^{-1}(\text{the centralizer of } \mathcal{M} \text{ in } SL(n, \mathbb{R})) = \text{the centralizer of } \varphi^{-1}(\mathcal{M}) \text{ in } \varphi^{-1}(SL(n, \mathbb{R}))$ is an analytic subgroup of L by Proposition 14. \square

Lemma 20 .

Let $n \geq 4$ and let \mathcal{A} be as in Lemma 19. The commutator subgroup of \mathcal{A} is $\mathcal{B} = \left\{ \begin{pmatrix} B & 0 \\ 0 & I_{n-2} \end{pmatrix} \mid B \in SL(2, \mathbb{R}) \right\}$.

If L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(\mathcal{B})$ is an analytic subgroup of L .

Proof:

That $[\mathcal{A}, \mathcal{A}] \subset \mathcal{B}$ is a simple computation. Equality follows from [4]. That $\varphi^{-1}(\mathcal{B})$ is an analytic subgroup of L follows from Lemma 19 and Lemma 4. \square

Lemma 21 .

Let \mathbb{F} be any field. The centralizer of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $SL(2, \mathbb{F})$ is $\left\{ \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \mid a, b \in \mathbb{F}, a^2 + c^2 = 1 \right\}$.

Proof:

This follows from a simple computation. \square

Corollary 22 .

The rotation group $SO(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$ is maximal abelian in $SL(2, \mathbb{R})$. In fact

$SO(2, \mathbb{R})$ is the centralizer of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $SL(2, \mathbb{R})$.

Proof:

$SO(2, \mathbb{R})$ is abelian and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SO(2, \mathbb{R})$. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in the centralizer of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $SL(2, \mathbb{R})$. Lemma 21 implies that $B = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}$ with $a^2 + c^2 = 1$. Hence $a = \cos(\theta)$ and $c = \sin(\theta)$ for some choice of $\theta \in \mathbb{R}$. Therefore $B \in SO(2, \mathbb{R})$ and $SO(2, \mathbb{R})$ is a maximal abelian subgroup of $SL(2, \mathbb{R})$. \square

Lemma 23 .

If $n \geq 2$, then the group $\mathcal{R} = \left\{ \begin{pmatrix} R & 0 \\ 0 & I_{n-2} \end{pmatrix} \mid R \in SO(2, \mathbb{R}) \right\}$ is maximal abelian in

$\mathcal{B} = \left\{ \begin{pmatrix} B & 0 \\ 0 & I_{n-2} \end{pmatrix} \mid B \in SL(2, \mathbb{R}) \right\}$. If L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(\mathcal{R})$ is an analytic subgroup of L .

Proof:

Corollary 22 implies that \mathcal{R} is a maximal abelian subgroup in \mathcal{B} if $n \geq 2$. If $n \geq 4$ then $\varphi^{-1}(\mathcal{B})$ is an analytic subgroup of L by Lemma 20. If $n = 2$ then $\varphi^{-1}(\mathcal{B})$ is an analytic subgroup of L by Proposition 5. If $n = 3$ then $\varphi^{-1}(\mathcal{B})$ is an analytic subgroup of L by Lemma 8. In every case $\varphi^{-1}(\mathcal{R})$ is maximal abelian in $\varphi^{-1}(\mathcal{B})$, therefore closed in $\varphi^{-1}(\mathcal{B})$ and hence is an analytic subgroup of L since $\varphi^{-1}(\mathcal{B})$ is an analytic subgroup of L . \square

Lemma 24

Let $H \subset SO(n, \mathbb{R})$ be the group generated by elements of the form $\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & 0 \\ & & & R(\theta) & & \\ & 0 & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}$, where

$R(\theta)$ is a 2-dimensional rotation. Then H acts transitively on S^{n-1} (and therefore, of course, $SO(n, \mathbb{R})$ acts transitively on S^{n-1}).

Proof:

If $x = (x_1, x_2) \neq (0, 0)$ and $\|x\| = 1$, there exists a 2-dimensional rotation matrix $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in SO(2, \mathbb{R})$ such that $R_\theta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for some θ .

Next, if $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ satisfies $\|x\| = 1$, then there exists $R(\theta_1) \in SO(2, \mathbb{R})$ such that $\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & 0 \\ & & & \ddots & & \\ & 0 & & & 1 & \\ & & & & & R(\theta_1) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x'_{n-1} \\ 0 \end{pmatrix} \in S^{n-1}$. Furthermore there exists $R(\theta_2) \in SO(2, \mathbb{R})$

such that
$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & 0 \\ & & & \ddots & & \\ & 0 & & & 1 & \\ & & & & & R(\theta_2) \\ & & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x'_{n-1} \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-3} \\ x'_{n-2} \\ 0 \\ 0 \end{pmatrix}.$$
 Continuing in this way, there exists a
$$R(\theta_{n-1}) \in SO(2, \mathbb{R}) \text{ such that } \begin{pmatrix} R(\theta_{n-1}) & & & & & \\ & \ddots & & & & \\ & & 1 & & & 0 \\ & & & \ddots & & \\ & 0 & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x'_2 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \in S^{n-1}.$$

Therefore H acts transitively on S^{n-1} (and, therefore, $SO(n, \mathbb{R})$ acts transitively on S^{n-1}). \square

Corollary 25

Identify $SO(n-1, \mathbb{R})$ with elements of $SO(n, \mathbb{R})$ of the form $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \subseteq SO(n, \mathbb{R})$ where $A \in SO(n-1, \mathbb{R})$.

Then $S^{n-1} \simeq SO(n, \mathbb{R})/SO(n-1, \mathbb{R})$ as $SO(n, \mathbb{R})$ spaces.

Proof:

This basic fact is well known and can easily be proved using the fact that $SO(n, \mathbb{R})$ acts transitively on S^{n-1} and that $SO(n-1, \mathbb{R})$ can be identified with the stability group of the north pole. \square

The next theorem is surely known, but the following proof might be of some interest.

Theorem 26

$SO(n, \mathbb{R})$ is algebraically generated by elements of the form
$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & 0 \\ & & & R(\theta) & & \\ & 0 & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}, \text{ where } R(\theta) \in$$

$SO(2, \mathbb{R})$.

Proof:

Let H be the subgroup of $SO(n, \mathbb{R})$ generated by the elements of the form

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & 0 \\ & & & R(\theta) & & \\ & 0 & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}, \text{ where } R(\theta) \in SO(2, \mathbb{R}). \text{ If } n = 2, \text{ then } H = SO(2, \mathbb{R}). SO(n-1, \mathbb{R}) \subset H$$

by induction. H and $SO(n, \mathbb{R})$ act transitively on S^{n-1} by Lemma 24. $S^{n-1} \simeq SO(n, \mathbb{R})/SO(n-1, \mathbb{R})$ as $SO(n, \mathbb{R})$ spaces by Corollary 25. Therefore $H = SO(n, \mathbb{R})$ by Lemma 13. \square

Corollary 27

If L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(SO(n, \mathbb{R}))$ is an analytic subgroup of L .

Proof:

$$\text{Let } H \text{ be the subgroup of } SO(n, \mathbb{R}) \text{ generated by the elements of the form } \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & 0 \\ & & & R(\theta) & & \\ & 0 & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix},$$

where $R(\theta) \in SO(2, \mathbb{R})$. $H = SO(n, \mathbb{R})$ by Theorem 26.

Let \mathcal{R} be defined as in Lemma 23. $\varphi^{-1}(\mathcal{R})$ is an analytic subgroup of L . Let R_{ij} be the result of swapping rows i and j in I_n . $\det(R_{ij}) = -1$, $R_{ij}^2 = I_n$, $R_{ij}^{-1} = R_{ij}$ and $R_{ij} \in O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid AA^T = A^T A = I_n\}$, the orthogonal group. Notice that the product of any two R_{ij} 's is an element of $SO(n, \mathbb{R})$. Let \mathcal{P} be the finite set of all products of pairs of the R_{ij} 's. It is an elementary computation that the generators for H are all contained in the finite set of conjugates of \mathcal{R} by elements of \mathcal{P} . φ^{-1} of any of these conjugates is a conjugate of $\varphi^{-1}(\mathcal{R})$ and is therefore an analytic subgroup of L . Hence, $\varphi^{-1}(SO(n, \mathbb{R})) = \varphi^{-1}(H)$ is an analytic subgroup of L by Lemma 4. \square

5 $\mathbb{R}^n \rtimes G(n)$ is an Algebraically Determined Polish Group

This section is devoted to showing that $\mathbb{R}^n \rtimes G(n)$ is an algebraically determined Polish group, thereby proving Theorem 1. This is done by verifying the hypotheses of Theorem 2.

The following proposition is a slight generalization of Lemma 25 of [2] with a different proof.

Proposition 28 .

Let \mathcal{K} be real or complex inner product space with $\dim(\mathcal{K}) \geq 2$ if \mathcal{K} is real or $\dim(\mathcal{K}) \geq 1$ if \mathcal{K} is complex. Let $x \in \mathcal{K}$ with $\|x\| \leq 2$. Then there exist $y, z \in \mathcal{K}$ with $\|y\| = \|z\| = 1$ and $y + z = x$.

Proof:

It suffices to consider the case in which \mathcal{K} is an inner product space over the reals, for if \mathcal{K} is complex, just restrict the scalars to the reals and the real dimension of \mathcal{K} will be at least two.

Let $x \in \mathcal{K}$ where \mathcal{K} is real and $\|x\| \leq 2$. Choose a unit vector v orthogonal to x and take $y = \frac{x+av}{2}$ and $z = \frac{x-av}{2}$, where $a = (4 - \|x\|^2)^{\frac{1}{2}}$. Clearly $y + z = x$ and $a^2 = (4 - \|x\|^2)$. Therefore $\|y\|^2 = \|\frac{x+av}{2}\|^2 = \frac{1}{4}(\|x\|^2 + a^2\|v\|^2) = \frac{1}{4}(\|x\|^2 + (4 - \|x\|^2)) = 1$. Thus $\|y\| = 1$. Similarly one can get $\|z\| = 1$. \square

Proposition 29 .

Let $\delta > 0$ and let $\mathcal{C} = \{(x, I_n) \mid \|x\| \leq \delta, x \in \mathbb{R}^n\}$. If L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi^{-1}(\mathcal{C})$ is analytic in $\varphi^{-1}(\mathbb{R}^n)$.

Proof:

The statement of this proposition makes sense because $\varphi^{-1}(\mathbb{R}^n)$ is a Polish group by Lemma 3.

Fix $x_0 \in \mathbb{R}^n$ with $\|x_0\| = \delta/2$. The mapping $\varphi^{-1}(SO(n, \mathbb{R})) \mapsto \varphi^{-1}(\mathbb{R}^n)$,

$$\begin{aligned} \varphi^{-1}((0, A)) &\mapsto \varphi^{-1}((0, A))\varphi^{-1}((x_0, I_n))\varphi^{-1}((0, A)^{-1}) \\ &= \varphi^{-1}((A(x_0), I_n)) \end{aligned}$$

is continuous on $\varphi^{-1}(SO(n, \mathbb{R}))$. Therefore the range of this mapping, namely $\varphi^{-1}(\{(x, I_n) \mid \|x\| = \delta/2\})$, is an analytic set in $\varphi^{-1}(\mathbb{R}^n)$ since $\varphi^{-1}(SO(n, \mathbb{R}))$ is an analytic subgroup of L by Corollary 27.

$\varphi^{-1}(\{(x, I_n) \mid \|x\| = \delta/2\}) \times \varphi^{-1}(\{(y, I_n) \mid \|y\| = \delta/2\}) \subset \varphi^{-1}(\mathbb{R}^n) \times \varphi^{-1}(\mathbb{R}^n)$ is also analytic since the product of two analytic sets is analytic. Therefore the mapping

$$\begin{aligned} \varphi^{-1}((x, I_n)) \times \varphi^{-1}((y, I_n)) &\mapsto \varphi^{-1}((x, I_n)) \cdot \varphi^{-1}((y, I_n)) = \varphi^{-1}((x + y, I_n)) \\ \varphi^{-1}(\{(x, I_n) : \|x\| = \delta/2\}) \times \varphi^{-1}(\{(y, I_n) : \|y\| = \delta/2\}) &\mapsto \varphi^{-1}(\mathbb{R}^n) \end{aligned}$$

is continuous and therefore has analytic range. But the range of this mapping is $\varphi^{-1}(\{(z, I_n) \mid \|z\| \leq \delta\})$ by Proposition 28. \square

Proposition 30 .

If L is a Polish group and $\varphi : L \mapsto \mathbb{R}^n \rtimes G(n)$ is an algebraic isomorphism, then $\varphi|_{\varphi^{-1}(\mathbb{R}^n)}$ is measurable with respect to $\mathcal{BP}(\varphi^{-1}(\mathbb{R}^n))$.

Proof:

The statement of this proposition makes sense because $\varphi^{-1}(\mathbb{R}^n)$ is a Polish group by Lemma 3.

Let $B_{\mathbb{R}^n}((0, I_n), \delta) = \{(x, I_n) \mid \|x\| < \delta\}$. Then $\varphi^{-1}(B_{\mathbb{R}^n}((0, I_n), \delta)) = \cup_{\ell \geq 1} \varphi^{-1}(\{(x, I_n) \mid \|x\| \leq \delta - \frac{1}{\ell}\})$ is analytic in $\varphi^{-1}(\mathbb{R}^n)$ by Proposition 29.

Fix $x_0 \in \mathbb{R}^n$. Then $\varphi^{-1}(\{(x, I_n) \mid \|x - x_0\| < \delta\}) = \varphi^{-1}((x_0, I_n)) \cdot \varphi^{-1}(\{(x, I_n) \mid \|x\| < \delta\})$ is analytic since $w \mapsto \varphi^{-1}((x_0, I_n)) \cdot w$ is continuous on $\varphi^{-1}(\mathbb{R}^n)$.

Let \mathcal{O} be open in \mathbb{R}^n such that $\mathcal{O} = \bigcup_{\ell \geq 1} B_{\mathbb{R}^n}((x_\ell, I_n), \delta_\ell) = \bigcup_{\ell \geq 1} \{(x, I_n) \mid \|x - x_\ell\| < \delta_\ell\}$. Then $\varphi^{-1}(\mathcal{O}) = \bigcup_{\ell \geq 1} \varphi^{-1}(\{(x, I_n) \mid \|x - x_\ell\| < \delta_\ell\})$ is analytic in $\varphi^{-1}(\mathbb{R}^n)$ since a union of a sequence of analytic sets is an analytic set. Hence $\varphi|_{\varphi^{-1}(\mathbb{R}^n)}$ is measurable with respect to $\mathcal{BP}(\varphi^{-1}(\mathbb{R}^n))$ since analytic sets have the Baire property. \square

Proof of Theorem 1:

The natural semidirect product $\mathbb{R}^n \rtimes G(n)$ is a Polish group. $\varphi^{-1}(\mathbb{R}^n)$ is closed in L by Lemma 3. $\varphi^{-1}(G(n))$ is analytic in L in every case by Proposition 5, by Proposition 14 and by Proposition 17. The natural θ in every case is always an injection. $\varphi|_{\varphi^{-1}(\mathbb{R}^n)}$ is measurable with respect to $\mathcal{BP}(\varphi^{-1}(\mathbb{R}^n))$ by Proposition 30. Theorem 2 now implies that $\mathbb{R}^n \rtimes G(n)$ is an algebraically determined Polish group. \square

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